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On the extremality of the action integral

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Abstract. Some necessary and sufficient conditions for a critical point of the action integral to be locally or globally extremal are proved. Applications to systems with finite or infinite number of degrees of freedom are discussed.

1. Introduction

The distinction between *stationary* and *extreme* is clearly stated in most of the mathematical literature (Carathéodory 1967, Lanczos 1970, Oden and Reddy 1976, Vainberg 1964). A functional is called stationary if its first-order variation around some given function vanishes: the function is called in this case a *critical point* of the functional. The word *extremal* is used to denote functions that render a functional extreme, i.e. either maximum or minimum.

The situation is quite different in classical mechanics and field theory (and also in some mathematical books). Many authors (e.g., Arnold 1979, Barut 1964, Feynman and Hibbs 1965, Morse 1934, Rosen 1969, Schulman 1981) call extremal, extreme or extremum either the function for which a functional has vanishing first-order variation or the functional itself. *Vice versa*, the word stationary has been used (Lindsay and Margenau 1957) in the sense of either maximum or minimum.

On top of that, it is common to speak of the *least* or *minimum action principle*. The name could be justified by historical reasons, but not when the principle is stated formally as meaning that 'the action integral is to be minimum over the physical trajectories', or at most with the warning that, actually, the action can be either minimum or maximum (Bradbury 1968, Goldstein 1950, Lindsay and Margenau 1957, Spiegel 1967, Wells 1967). As many examples in this paper show, in general the action integral has neither a minimum nor a maximum (even locally) over the trajectories. The source of this confusion can perhaps be traced to the philosophical motivations of the principle of least action of Maupertuis, as a manifestation of the 'economy of nature' (Lindsay and Margenau 1957, Yourgrau and Mandelstam 1968). These metaphysical concomitances have been completely disallowed (Born 1969), but they persist at least at the terminological level.

Actually in the derivation of the Euler–Lagrange equations through Hamilton's variational principle, only the stationarity of the action integral is needed. On the contrary, in other problems like the search for optimal solutions in the theory of control and optimisation (Bellman 1967, Pontryagin *et al* 1964), or the approximation of solutions by the Rayleigh–Ritz method (Helleman 1978), the extremality is essential.

Necessary and sufficient conditions for a functional to have an extremum can be found in the monographs on variational calculus and functional analysis (e.g. Berger 1977, Carathéodory 1967, Funk 1962, Gelfand and Fomin 1963, Lanczos 1970, Lippmann 1972, Morse 1934, Oden and Reddy 1976, Pars 1962, Vainberg 1964), but they are either restricted to the simplest cases (finite number of degrees of freedom, no dependence on second- and higher-order derivatives, etc) or are quite abstract and do not examine explicitly the examples commonly found in field theory. In fact, the regularity properties of minima of multiple integrals remain only partially proven, as remarked by Berger (1977). An outstanding and actual example is provided by the classical non-abelian pure Yang–Mills theories (Eguchi *et al* 1980, Actor 1979) in the Euclidean framework. Whereas the minimal nature of the action integral is simply proved for selfdual or anti-selfdual solutions (Belavin *et al* 1975), to our knowledge the question is open in the general case, unless particular gauge groups (such as SU(2) or SU(3)) are assumed and solutions extensible to the one-point compactification of $\mathbb{R}^4$ are considered (Bourguignon *et al* 1979, Bourguignon and Lawson 1981). Ellipticity of the associated complex bundle is essential for the proof of these results. (See also Jaffe (1982) for a recent discussion, and Taubes (1982) for the non-minimality of some Yang–Mills–Higgs solutions.)

For these reasons we think that a discussion of some necessary and sufficient conditions for the existence of extrema of a functional, valid for systems with finite or infinite number of degrees of freedom and with dependence on arbitrarily high derivatives, will be useful. We have applied these criteria to some of the problems frequently found in mechanics and field theory, in order to show their power and ease of application and at the same time to dispel possible misconceptions (such as a belief in the minimal character of the solutions in field theory if the interval in the independent variables is small enough). Of course, the applications can be extended to any extremal problems, like the ones of control and optimisation. To keep the work within reasonable bounds we do not consider the inclusion of constraints.

The plan of this paper is as follows.

In § 2 we define the notation and summarise the concepts that will be used through the rest of the paper.

In § 3 we prove three criteria for minimality. The first one is a set of algebraic necessary conditions. The second one shows how the concept of *strong ellipticity* (also an algebraic condition) is sufficient to ensure that the action integral is locally strictly minimum. Finally, the third criterion, based on the positivity of the spectrum of strongly elliptic differential operators, gives necessary and sufficient conditions for extremality.

In § 4 we show some applications of the criteria of § 3. First we display examples in classical mechanics (relativistic particle in a given electromagnetic field) and field theory (interacting scalar and electromagnetic field, classical electrodynamics, Dirac equation, time-dependent Schrödinger equation) in which the action integral is never extreme, even locally. Then we consider some discrete and continuous systems where the action is strictly minimum, at least locally. The extremum character of the action in the large is considered in the next group of examples: in some cases (harmonic and anharmonic oscillators, particle in a constant magnetic field or in a Coulomb potential) the action is never globally extreme; in others (a double well potential) some trajectories are globally minimal, while others are not.

Finally, in § 5 we show how, given any equation (or systems of equations), it is always possible to find a generalised Lagrangian such that any solution of the equation is a globally minimal point of the corresponding action integral (in fact the action reaches its absolute minimum at these solutions). From the numerical point of view the associated variational search for solutions may however face serious difficulties, some of which are briefly suggested therein.

2. Notation and definition

Independent variables (such as the time t in ordinary non-relativistic mechanics, the invariant interval s in relativistic mechanics, the coordinates x^0, x^1, x^2, x^3 of an event in Minkowskian field theory, etc) will be denoted by $x \equiv (x^1, \ldots, x^N)$. The symbol $u \equiv (u^1, \ldots, u^R)$ will stand for real dependent variables (the generalised coordinates q_i of a discrete mechanical system, the fields in continuous systems, etc).

A set of non-negative integers $(a_1, \ldots, a_N)$ constitutes a multi-index $a \equiv (a_1, \ldots, a_N)$, of order $|a| \equiv a_1 + \ldots + a_N$.

The symbol $u^i_{,a}(x)$ will represent as usual the partial derivative

$$u^i_{,a}(x) \equiv \partial^{|a|} u^i(x)/(\partial x^1)^{a_1} \ldots (\partial x^N)^{a_N}. \tag{2.1}$$

By $\mathscr{L}[x, u]$ we shall denote a sufficiently smooth (say C^∞) function of x, u and the derivatives $u_{,a}$ up to some finite order $|a| \leqslant l$, $l > 0$. In our applications, $\mathscr{L}[x, u]$ will be the Lagrangian density.

Given a domain (bounded open set) $\Omega \subset \mathbb{R}^N$, we shall define a functional $A_\Omega[u]$ (the action integral) as

$$A_\Omega[u] \equiv \int_\Omega \mathrm{d}x \, \mathscr{L}[x, u(x)]. \tag{2.2}$$

Such a functional exists and is C^∞ if $u \in C^l(\bar{\Omega})$, the Banach space of the vector-valued real functions $u(x)$ such that $u^i_{,a}$ is continuous on the closure $\bar{\Omega}$ for $|a| \leqslant l$, with norm $\|u\|_{C^l(\bar{\Omega})} \equiv \Sigma_{|a| \leqslant l} \sup_{i, \bar{\Omega}} |u^i_{,a}(x)|$ (Berger 1977). In fact, $A_\Omega[u]$ has the Taylor expansion

$$A_\Omega[u+v] = \sum_{j=0}^{J} \frac{1}{j!} A_\Omega^{(j)}[u](\underbrace{v, \ldots, v}_{j}) + R_\Omega^{(J+1)}[u, v] \tag{2.3}$$

where $A_\Omega^{(j)}[u]$ is the functional (Fréchet) jth derivative of A_Ω at u, and the remainder $R_\Omega^{(J+1)}$ satisfies $|R_\Omega^{(J+1)}[u, v]| = \mathrm{o}(\|v\|^J_{C^l(\bar{\Omega})})$.

In particular the first terms in (2.3) are

$$A_\Omega^{(0)}[u] = A_\Omega[u], \tag{2.4a}$$

$$A_\Omega^{(1)}[u](v) = \sum_{r,a} \int_\Omega \mathrm{d}x \frac{\partial \mathscr{L}}{\partial u^r_{,a}} v^r_{,a}, \tag{2.4b}$$

$$A_\Omega^{(2)}[u](v, v) = \sum_{r,s,a,b} \int_\Omega \mathrm{d}x \frac{\partial^2 \mathscr{L}}{\partial u^r_{,a} \, \partial u^s_{,b}} v^r_{,a} v^s_{,b}. \tag{2.4c}$$

We are now in a position to define precisely what we shall understand by the terms stationary, extreme and minimum applied to the integral action.

Definition 1. A_Ω is *stationary* at u_0 if

$$A_\Omega^{(1)}[u_0](v) = 0 \qquad \forall v \in C^l_0(\Omega) \tag{2.5}$$

(where $C_0^l(\Omega)$ is the subspace of those functions in $C^l(\Omega)$ with compact support in the open set Ω; such functions and their derivatives up to order l vanish on the boundary $\partial\Omega$ of Ω).

In the sequel we shall write u_c to denote an arbitrary *critical* point of A_Ω (i.e. A_Ω stationary at u_c).

When $u \in C^{2l}(\bar{\Omega})$, a simple integration by parts yields

$$A_\Omega^{(1)}[u](v) = \sum_r \int_\Omega \mathrm{d}x\, v^r \frac{\delta \mathscr{L}}{\delta u^r} \tag{2.6}$$

where the variational derivative is

$$\frac{\delta \mathscr{L}}{\delta u^r} \equiv \sum_a (-1)^{|a|} D_a \frac{\partial \mathscr{L}}{\partial u^r_{,a}} \tag{2.7}$$

with $D_a \equiv (D_1)^{a_1} \ldots (D_N)^{a_N}$. The symbol D_i denotes the total derivative with respect to x^i:

$$D_i \equiv \frac{\partial}{\partial x^i} + \sum_{r,a} u^r_{,a+e_i} \frac{\partial}{\partial u^r_{,a}} \tag{2.8}$$

where $a + e_i$ is the multi-index $(a_1, \ldots, a_i + 1, \ldots, a_N)$. Therefore, we have the well known result:

Proposition 1. A_Ω is stationary at $u_c \in C^{2l}(\bar{\Omega})$ iff

$$(\delta \mathscr{L}/\delta u^r)[x, u_c(x)] = 0, \qquad r = 1, \ldots, R, \tag{2.9}$$

i.e. iff $u_c(x)$ satisfies the Euler–Lagrange equations.

Definition 2. A_Ω has a *minimum* at u_0 if $\exists$ a ball $B_\varepsilon(\Omega) \equiv \{v \in C_0^l(\Omega): \|v\|_{C^l(\bar{\Omega})} \leqslant \varepsilon\}$, $\varepsilon > 0$, such that

$$A_\Omega[u_0 + v] \geqslant A_\Omega[u_0] \qquad \forall v \in B_\varepsilon(\Omega). \tag{2.10}$$

Proposition 2. The action integral A_Ω has a minimum at u_0 only if A is stationary at u_0 and

$$A_\Omega^{(2)}[u_0](v, v) \geqslant 0 \qquad \forall v \in C_0^l(\Omega). \tag{2.11}$$

(The proof follows immediately from (2.3).)

Definition 3. A has a *strict minimum* at u_c if $\exists B_\varepsilon(\Omega)$:

$$A_\Omega[u_c + v] > A_\Omega[u_c] \qquad \forall 0 \neq v \in B_\varepsilon(\Omega). \tag{2.12}$$

Proposition 3. A sufficient condition for A_Ω to have a strict minimum at u_c is

$$A_\Omega^{(2)}[u_c](v, v) > 0 \qquad \forall 0 \neq v \in C_0^l(\Omega). \tag{2.13}$$

(The proof is again straightforward.)

Remark 1. The dual notions of maximum and strict maximum are defined *mutatis mutandis*, and the previous propositions carry over to this case with the pertinent trivial modifications. When A_Ω has either a minimum or a maximum at u_c we shall say that A_Ω has an *extremum* at u_c.

Definition 4. A critical point u_c will be called *locally minimal* at x_0 if $\exists$ a domain $\Omega_0 \ni x_c$ such that A_{Ω_0} has a minimum at u_c. When u_c is locally minimal at each $x \in \Omega$, we shall say that A_Ω is *locally minimum* at u_c.

Note that this requirement is weaker than the condition (2.10) in definition 2, since now the inequality (2.10) is only supposed to hold for v in suitable x-dependent balls. Actually, A_Ω minimum at $u_c \Rightarrow A_\Omega$ locally minimum at u_c. In § 4.3 several examples will illustrate how the action integral A_Ω can be locally minimum at any u_c and every Ω whilst A_Ω will not in general be minimum if Ω is large enough.

Definition 5. A critical point u_c will be called *globally minimal* if A_Ω has a minimum at u_c for all domains Ω. The action will accordingly be said to be *globally minimum* at u_c.

Remark 2. Note that 'local' and 'global' refer to the independent variables and not to the functional space; we are not comparing the values of A_Ω for different u_c's.

Remark 3. It should be noted that A_Ω minimum at $u_c \Leftrightarrow A_{\Omega'}$ minimum at u_c for every $\Omega' \subseteq \Omega$.

Remark 4. The addition of a divergence $D_i f^i[x, u]$ (with possible dependence on u and its derivatives up to order l) to the Lagrangian density does not change the difference $A[u+v]-A[u]$ for any $v \in C_0^l(\Omega)$. In fact, the new action integral is

$$\tilde{A}_\Omega[u] \equiv \int_\Omega \mathrm{d}x \{\mathscr{L}[x, u(x)] + D_i f^i[x, u(x)]\} = A_\Omega[u] + \omega_{\partial\Omega}[u] \tag{2.14}$$

where the functional $\omega_{\partial\Omega}[u]$ depends only on the values of u and its derivatives on the frontier $\partial\Omega$.

Then, $\omega_{\partial\Omega}[u+v] = \omega_{\partial\Omega}[u]$, $\forall v \in C_0^l(\Omega) \Rightarrow \tilde{A}_\Omega[u+v] = A_\Omega[u+v] + \omega_{\partial\Omega}[u]$, and therefore

$$\tilde{A}_\Omega[u+v] - \tilde{A}_\Omega[u] = A_\Omega[u+v] - A_\Omega[u]. \tag{2.15}$$

In particular, $\tilde{A}_\Omega^{(j)} = A_\Omega^{(j)}$, $\forall j \geq 1$, and the critical, minimal, strictly minimal, etc character of a function $u(x)$ is unaffected by adding a divergence to the Lagrangian density.

3. Criteria for minima

Given a critical point $u_c \in C^\infty(\bar{\Omega})$ of A_Ω, let us write

$$A_{rs}^{ab}(x) \equiv (\partial^2 \mathscr{L} / \partial u^r_{,a}\, \partial u^s_{,b})[x, u_c(x)] \tag{3.1}$$

and consider the differential operator matrix $\tau(x, D)$ with entries

$$\tau_{rs}(x, D) \equiv \sum_{|a|,|b| \leq l} (-1)^{|a|} D_a A_{rs}^{ab}(x) D_b \tag{3.2}$$

and a weighted principal symbol $\bar{\tau}^n(x, D)$ defined, for a multi-index n, by

$$\bar{\tau}^n_{rs}(x, D) \equiv \sum_{n\cdot(a+b)=M_n} (-1)^{|a|} D_a A^{ab}_{rs}(x) D_b \tag{3.3}$$

where $M_n \equiv \max n \cdot (a+b)$ over those multi-indices a, b for which $A^{ab}_{..} \not\equiv 0$.

With the standard notation $\xi \cdot x \equiv \Sigma_1^N \xi_i x^i$, $(\lambda, \mu) \equiv \Sigma_1^R \lambda^{r*} \mu^r$ we claim:

Theorem 1. A necessary condition for minimum. A_Ω minimum at $u_c \in C^\infty(\bar{\Omega}) \Rightarrow$

$$(\lambda, \bar{\tau}^n(x, \mathrm{i}\xi)\lambda) \geqslant 0 \qquad \forall \xi \in \mathbb{R}^N, \lambda \in \mathbb{C}^R, x \in \Omega, n \in \mathbb{Z}^N_+. \tag{3.4}$$

Proof. After complexification of the symmetric bilinear functional $A^{(2)}_\Omega[u_c](v, w)$:

$$\begin{aligned} A^{(2)}[u_c](v_1 + \mathrm{i}v_2, w_1 + \mathrm{i}w_2) & \\ \equiv A^{(2)}[u_c](v_1, w_1) + A^{(2)}[u_c](v_2, w_2) & \\ + \mathrm{i}\{A^{(2)}[u_c](v_1, w_2) - A^{(2)}[u_c](v_2, w_1)\} & \end{aligned} \tag{3.5}$$

it is plain from proposition 2 that A_Ω minimum at u_c implies

$$A^{(2)}[u_c](\mathrm{e}^{\mathrm{i}\xi\cdot x}\eta, \mathrm{e}^{\mathrm{i}\xi\cdot x}\eta) \geqslant 0 \qquad \forall \eta \equiv \eta_1 + \mathrm{i}\eta_2, \eta_i \in C^l_0(\Omega), \xi \in \mathbb{R}^N. \tag{3.6}$$

The asymptotic behaviour of (3.6) when ξ increases to infinity as $(\rho^{n_1}\hat{\xi}^1, \ldots, \rho^{n_N}\hat{\xi}^N)$ forces the dominant part of (3.6) to be non-negative:

$$\int_\Omega \mathrm{d}x \sum_{r,s} \eta^{r*}(x) \bar{\tau}^n_{rs}(x, \mathrm{i}\xi) \eta^s(x) \geqslant 0. \tag{3.7}$$

The arbitrariness of η thus ensures (3.4), by a suitable sequential approach to limiting delta-functions.

Definition 6. $\tau(x, D)$ is called *strongly elliptic* in Ω if $\exists k > 0$ such that

$$(\lambda, \bar{\tau}(x, \mathrm{i}\xi)\lambda) \geqslant k|\lambda|^2|\xi|^{2l} \qquad \forall x \in \Omega, \lambda \in \mathbb{C}^R, \xi \in \mathbb{R}^N, \tag{3.8}$$

where $\bar{\tau}(x, D)$ stands for the principal symbol (3.3) with $n = (1, \ldots, 1)$.

The following important theorem is adapted from well known results in the minimisation theory of critical points (Berger 1977).

Theorem 2. A sufficient condition for locally minimum. $\tau(x, D)$ strongly elliptic in $\Omega \Rightarrow A_\Omega$ is locally strictly minimum at u_c.

Proof. We shall follow closely the arguments in Berger (1977). The strong ellipticity ensures the important Gårding inequality: $\exists k_1 \in \mathbb{R}$, $K > 0$ such that

$$(v, (\tau + k_1)v)_{L^2(\Omega)} \geqslant K\|v\|^2_{H^l(\Omega)} \qquad \forall v \in C^l_0(\Omega) \tag{3.9}$$

where $(\cdot, \cdot)_{L^2}$ stands for the usual scalar product in the Hilbert space $\bigoplus_1^R L^2(\Omega, \mathrm{d}x)$, and $\|\cdot\|_{H^l}$ is the Sobolev norm defined by

$$\|v\|^2_{H^l(\Omega)} \equiv \sum_{r,|a|\leqslant l} \|v^r_{,a}\|^2_{L^2(\Omega)}. \tag{3.10}$$

For a given $x_0 \in \Omega$, we can always find a sufficiently small domain $\Omega_0 \ni x_0$ with the property

$$\|v\|^2_{L^2(\Omega_0)} \leqslant (K/2|k_1|)\|v\|^2_{H^l(\Omega_0)} \qquad \forall v \in C_0^\infty(\Omega_0). \tag{3.11}$$

It suffices to use Poincaré's inequality

$$\|v\|_{L^2(\Omega)} \leqslant k(\Omega)\|\nabla v\|_{L^2(\Omega)} \tag{3.12}$$

(valid for any domain Ω and $v \in C_0^\infty(\Omega)$) and a simple dilation argument.

From (3.9) and (3.11) we get

$$A^{(2)}_{\Omega_0}[u_c](v, v) \geqslant \tfrac{1}{2}K\|v\|^2_{H^l(\Omega_0)} \tag{3.13}$$

for every $v \in C_0^\infty(\Omega_0)$ and thus on $H_0^l(\Omega_0)$ by completion of $C_0^\infty(\Omega_0)$ in the H^l norm.

Proposition 3 shows now that A_{Ω_0} has a strict minimum at u_c, and the possible x_0-dependence of the domain Ω_0 renders it local.

Our next aim will be to provide a powerful operator-theoretic criterion for the existence of a strict minimum in a given domain Ω. To this end we need to recall (without proof) some mathematical results of spectral analysis. Suppose that $\tau(x, D)$ is strongly elliptic in the bounded open set Ω with smooth boundary $\partial\Omega$. Gårding's inequality (3.9) and the Lax–Milgram theorem (Berger 1977) ensure the existence of a *self-adjoint* operator $T_\Omega(\tau)$ (the Friedrichs extension of the restriction $\tau \restriction C_0^\infty(\Omega)$ of τ to $C_0^\infty(\Omega)$) with domain

$$D(T_\Omega(\tau)) = H_0^l(\Omega) \cap \{u \in L^2(\Omega)\colon \tau u \in L^2(\Omega)\} \tag{3.14}$$

and such that

$$T_\Omega(\tau)u \equiv \tau u \qquad \text{(in distribution sense).} \tag{3.15}$$

This operator $T_\Omega(\tau)$ enjoys the following properties (Dunford and Schwartz 1963):

(i) $D(T_\Omega(\tau)) = H_0^l(\Omega) \cap H^{2l}(\Omega)$;

(ii) it is semibounded, with discrete spectrum

$$\sigma(T_\Omega(\tau)) \equiv \{\mu_1(\Omega, \tau) \leqslant \mu_2(\Omega, \tau) \leqslant \ldots\};$$

(iii) it is the closure of the operator

$\tau \restriction \{u \in C^\infty(\bar{\Omega})$: the normal derivatives $\partial^j_{\text{normal}}\, u = 0, j = 0, \ldots, l-1$ on $\partial\Omega\}$;

(iv) its eigenfunctions lie in $C^\infty(\bar{\Omega})$.

Armed with these results, we claim:

Theorem 3. A sufficient condition for strict minimum. Let $\tau(x, D)$ be strongly elliptic, with $\partial\Omega$ smooth. Then

(a) $\mu_1(\Omega, \tau) > 0 \Rightarrow A_\Omega$ has a strict minimum at u_c;

(b) $\mu_1(\Omega, \tau) < 0 \Rightarrow A_\Omega$ has not an extremum at u_c.

Proof. Since by hypothesis $\mathscr{L}[x, u]$ is smooth and $u_c \in C^\infty(\bar{\Omega})$, the coefficients $A^{ab}_{rs}(x)$ given by (3.1) will be bounded on Ω and thus

$$|A^{(2)}_\Omega[u_c](\eta, \chi)| \leqslant \text{constant}\ \|\eta\|_{H^l(\Omega)}\|\chi\|_{H^l(\Omega)} \tag{3.16}$$

for $\forall \eta, \chi \in H^l(\Omega)$. We can thus evaluate $A_\Omega^{(2)}[u_c]\,(v, v)$ for any $v \in C_0^l(\Omega)$ as the limit of $A_\Omega^{(2)}[u_c](v_j, v_j)$, $j \to \infty$, over a sequence $C_0^\infty(\Omega) \ni v_j \to v$ in the $H^l(\Omega)$ norm.

(a) As

$$A_\Omega^{(2)}[u_c](v_j, v_j) = (v_j, T_\Omega(\tau)v_j) \geqslant \mu_1(\Omega, \tau)\|v_j\|^2_{L^2(\Omega)} \tag{3.17}$$

in the limit $j \to \infty$ we get

$$A_\Omega^{(2)}[u_c](v, v) \geqslant \mu_1(\Omega, \tau)\|v\|^2_{L^2(\Omega)} > 0 \qquad \text{if } 0 \neq v \in C_0^l(\Omega). \tag{3.18}$$

(b) Should A_Ω have an extremum at u_c then forcefully it should be a minimum because of (ii). But then proposition 2 would imply the positivity of the quadratic form $A_\Omega^{(2)}$ on $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$, and therefore the positivity of the operator $T_\Omega(\tau)$ associated with its Friedrichs extension (Reed and Simon 1975), in contradiction with hypothesis (b).

In view of the preceding results, the systematic analysis of the minimality of the action integral might roughly proceed along the following steps.

First, try to see whether some of the necessary conditions established in theorem 1 are violated, so precluding the minimum character. Otherwise, it may likely turn out that $\tau(x, D)$ is strongly elliptic, thus warranting locally a strict minimum through theorem 2. For Ω small enough, A_Ω will actually be a strict minimum (not only locally), because of Gårding's and Poincaré's inequalities. However, when Ω is enlarged, it might happen that the minimum character disappears even though A_Ω still has locally a minimum at u_c. The onset of this situation is marked by the lowest eigenvalue $\mu_1(\Omega, \tau)$ becoming negative, according to theorem 3.

4. Applications

This section will be devoted to illustrating how the previous criteria can be put to good use for the analysis of minimality of the action integral in several cases of physical interest. Among these there will appear some examples with complex-valued field functions in real-valued Lagrangians. The canonical way of dealing with them would be to decompose such variables into their real and imaginary parts. Such a procedure may be cumbersome, and thus it seems desirable to work directly with the fields themselves and their complex conjugates as if they were independent, but with complex-conjugate infinitesimal variations. This formal handling may be justified in most applications, provided that the expressions involving the fields are analytic (as we shall tacitly assume when the Lagrangian density is not given explicitly).

4.1. Applications of theorem 1: non-existence of minima

4.1.1. Charged relativistic particle in an external electromagnetic field

$$\mathscr{L}[s, z(s)] \equiv -\tfrac{1}{2} g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu - A_\mu(z)\dot{z}^\mu \tag{4.1}$$

where s is the invariant interval in Minkowski space and $\dot{z}^\mu \equiv \mathrm{d}z^\mu/\mathrm{d}s \equiv Dz^\mu$.

In this example there is only one independent variable, the interval s, and four dependent variables, the coordinates z^μ of the particle ($N = 1, R = 4$).

The principal symbol $\bar{\tau}(s, D)$ is the 4×4 matrix with components $Dg_{\mu\nu}D$. The matrix $-g_{\mu\nu}\xi^2$ is not definite (either positive or negative) and therefore according to theorem 1 (and its trivial counterpart for the case of maximum), the action is never minimum nor maximum: even for Δs arbitrarily short there are small variations around any given trajectory z_c for which the action integral decreases, and other variations for which it increases.

4.1.2. *Charged scalar and electromagnetic fields in interaction*

$$\mathcal{L} = \sum_{i=1}^{S} \mathcal{L}_{\phi^{(i)}} + \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{int}} \tag{4.2}$$

with

$$\mathcal{L}_{\phi^{(i)}} \equiv \{D_\mu \phi^{(i)}\}^* D^\mu \phi^{(i)} - m_i^2 |\phi^{(i)}|^2, \tag{4.3a}$$

$$\mathcal{L}_{\text{EM}} \equiv -\tfrac{1}{4}\{D_\mu A_\nu - D_\nu A_\mu\}\{D^\mu A^\nu - D^\nu A^\mu\} \tag{4.3b}$$

and $\mathcal{L}_{\text{int}}(\phi^{(1)}, \ldots, \phi^{(S)}, A)$ with at most linear dependence on first-order derivatives and no dependence on second- or higher-order derivatives.

In this example $x = \{x^\mu; \mu = 0, 1, 2, 3\}$, $u = \{\phi^{(i)}, \phi^{(i)*}, A^\mu;\ i = 1, \ldots, S,\ \mu = 0, \ldots, 3\}$, $N = 4$, $R = 2S + 4$.

If we denote by $\lambda^{(i)}$ the first S components of λ^r and by λ_A^μ the last four components, we have

$$(\lambda, \bar{\tau}(x, \mathrm{i}\xi)\lambda) = 2 \sum_{i=1}^{S} |\lambda^{(i)}|^2 \langle \xi, \xi \rangle - \{\langle \lambda_A, \lambda_A \rangle \langle \xi, \xi \rangle - |\langle \lambda_A, \xi \rangle|^2\} \tag{4.4}$$

where $\langle \alpha, \beta \rangle \equiv \alpha_\mu^* \beta^\mu$ is the Lorentz scalar product.

Again, the expectation value (4.4) has no definite sign and therefore the action integral is neither maximum nor minimum. This result applies in particular to the free electromagnetic field.

A similar argument proves that the action integral is never extreme in the case of the electromagnetic field interacting with an external current, $\mathcal{L} = \mathcal{L}_{\text{EM}} - A_\mu j^\mu$ (the interaction term gives no contribution to the principal symbol $\bar{\tau}$).

The inclusion in (4.2) of Dirac fields with non-derivative couplings would not change the principal symbol $\bar{\tau}(x, D)$, and the previous result still holds. In particular, for the usual classical electrodynamics (Dirac and Maxwell fields interacting through the minimal coupling), the action integral is never extreme.

4.1.3. *Dirac field with non-derivative self-coupling*

$$\mathcal{L} \equiv \tfrac{1}{2}\mathrm{i}\{\bar{\psi}\gamma_\mu D^\mu \psi - (\overline{D_\mu \psi})\gamma^\mu \psi\} + \mathcal{L}_{\text{int}}(\psi, \bar{\psi}). \tag{4.5}$$

In this example

$$(\lambda, \bar{\tau}(x, \mathrm{i}\xi)\lambda) = -2\bar{\lambda}\xi_\mu \gamma^\mu \lambda \tag{4.6}$$

which once more is indefinite, and thus the action integral is neither minimum nor maximum.

4.1.4. *Schrödinger equation*

$$\mathcal{L} \equiv \tfrac{1}{2}\mathrm{i}(\psi^* \, \partial\psi/\partial t - (\partial\psi^*/\partial t)\,\psi) - |\nabla\psi|^2 - V(x, t)|\psi|^2. \tag{4.7}$$

Now,

$$\tau(x, D) = 2\{\mathrm{i}\,\partial/\partial t + \nabla \cdot \nabla - V\} \tag{4.8}$$

and therefore if we denote by $\xi_{(0)}$ the component of ξ_i corresponding to the time variable and we choose a multi-index $n = (1, 0, 0, 0)$ (with the time written in the first place), the corresponding weighted principal symbol gives

$$(\lambda, \bar{\tau}^n(x, \mathrm{i}\xi)\lambda) = -2|\lambda|^2\xi_{(0)} \tag{4.9}$$

which has no definite sign. Therefore the Schrödinger action is neither minimum nor maximum.

4.2. *Applications of theorem 2: actions which have locally a strict minimum*

4.2.1. *Non-relativistic mechanical system*

$$\mathscr{L} \equiv \tfrac{1}{2}\dot{q}^r m_{rs}\dot{q}^s + \mathscr{L}_{\text{int}} \tag{4.10}$$

where $\dot{q}^r \equiv \mathrm{d}q^r/\mathrm{d}t$, the 'mass matrix' $m \equiv (m_{rs})$ is symmetrical and strictly positive definite, and $\mathscr{L}_{\text{int}}$ depends smoothly on t, q, and at most linearly on $\dot{q}$.

In this example, the independent variable x is just the time variable t, and the dependent variables are $u \equiv \{q^r, r = 1, \ldots, R\}$. The components of the principal symbol $\bar{\tau}$ are

$$\bar{\tau}_{rs}(t, \mathrm{i}\xi) = m_{rs}\xi^2. \tag{4.11}$$

Then $\tau(x, D)$ is strongly elliptic over the whole time interval $\mathbb{R}$ and according to theorem 2, the action integral A_Ω will be locally strictly minimum at any q_c smooth on $\bar{\Omega}$.

4.2.2. *Euclidean scalar field*

$$\mathscr{L} \equiv \tfrac{1}{2}(D_i\phi)\delta^{ij}D_j\phi + \mathscr{L}_{\text{int}} \tag{4.12}$$

where ϕ is real valued (the generalisation to more than one field or/and to the complex case is trivial), and $\mathscr{L}_{\text{int}}$ depends smoothly on $x = \{x^i, i = 1, \ldots, N\}$, ϕ and at most linearly on $D_i\phi$.

Now

$$\bar{\tau}(x, \mathrm{i}\xi) = \xi_j\delta^{jk}\xi_k \equiv \xi^2. \tag{4.13}$$

Therefore $\tau(x, D)$ strongly elliptic over $\mathbb{R}^N \Rightarrow$ the action A_Ω is locally strictly minimum at any ϕ_c smooth on $\bar{\Omega}$.

4.3. *Applications of theorem 3*

4.3.1. *Harmonic oscillator in R dimensions*

$$\mathscr{L} \equiv \tfrac{1}{2}\sum_{r=1}^{R} \{(\dot{q}^r)^2 - \omega_r^2(q^r)^2\}, \qquad \omega_r \geq 0, r = 1, \ldots, R. \tag{4.14}$$

This is a particular case of § 4.2.1 and therefore the action integral A_Ω will be locally strictly minimum at any q_c and every domain Ω. To elucidate the size of Ω for which A is strictly minimum at q_c, it will suffice to compute the lowest eigenvalue of $T_\Omega(\tau)$. This operator is given by

$$T_\Omega(\tau) = \delta_{rs}\{-D^2 - \omega_r^2\} \tag{4.15}$$

with Dirichlet boundary conditions on $\partial\Omega$. Therefore $T_\Omega(\tau)$ is just a direct sum of one-dimensional free Schrödinger operators in an infinite square well potential of width $|\Omega| = t_2 - t_1$, if $\Omega =]t_1, t_2[$. Consequently

$$\mu_1(\Omega, \tau) = (\pi/|\Omega|)^2 - \omega^2, \qquad \omega \equiv \sup_r \omega_r. \tag{4.16}$$

If T denotes the span, $2\pi/\omega$, then theorem 3 leads immediately to the following conclusions:

(i) $|\Omega| < T/2 \Rightarrow A_\Omega$ has a strict minimum at any q_c;

(ii) $|\Omega| > T/2 \Rightarrow A_\Omega$ has not an extremum at any q_c.

The border case $|\Omega| = T/2$ is not covered by this theorem. To ascertain what happens in this situation, we must resort to higher-order variations. However, $\mathscr{L}$ being quadratic, $A_\Omega^{(j)}[q_c] = 0$, $\forall j \geq 3$ and therefore A_Ω has a (non-strict) minimum at every q_c.

It should be noted finally that, due to the vanishing of the higher variations $A_\Omega^{(j)}$, $j \geq 3$, the action integral has a strict minimum at q_c not only under infinitesimal variations but also for arbitrary large variations vanishing at the end points of q_c, so that given (t_1, q_1), (t_2, q_2) there is only one physical trajectory q_c joining them whenever $|t_2 - t_1| < T/2$. If $|t_2 - t_1| = T/2$, the action integral A_Ω is still minimum at q_c, but there is now a continuum of physical trajectories, with the same value of the action, connecting the end points. These are conjugate in Jacobi's sense (Lippmann 1972, Schulman 1981).

4.3.2. Non-relativistic charged particle in an external electromagnetic field

$$\mathscr{L} \equiv \tfrac{1}{2} m \dot{\boldsymbol{q}}^2 + (e/c)\dot{\boldsymbol{q}} \cdot \boldsymbol{A}(\boldsymbol{q}, t) - e\phi(\boldsymbol{q}, t). \tag{4.17}$$

This is again a particular instance of § 4.2.1 and therefore the result obtained there applies: the action A_Ω has locally a strict minimum at any q_c smooth on $\bar{\Omega}$.

To strengthen this conclusion we need the explicit form of the operator $\tau(t, D)$:

$$\begin{aligned} \tau_{ij}(t, D) &= \left[-m\delta_{ij}D^2 - \frac{e}{c} D \frac{\partial A_i}{\partial q^j} + \frac{e}{c}\frac{\partial A_j}{\partial q^i} D + \frac{e}{c}\frac{\partial^2 A_l}{\partial q^i\, \partial q^j}\dot{q}^l - e\frac{\partial^2 \phi}{\partial q^i\, \partial q^j}\right]_{q=q_c} \\ &= \left[-m\delta_{ij}D^2 + \frac{e}{2c}\epsilon_{ijk}(B^k D + DB^k) + \frac{e}{2}\left(\frac{\partial E_i}{\partial q^j} + \frac{\partial E_j}{\partial q^i}\right)\right. \\ &\quad \left. + \frac{e}{2c}\dot{q}^l\left(\epsilon_{ilk}\frac{\partial B^k}{\partial q^j} + \epsilon_{jlk}\frac{\partial B^k}{\partial q^i}\right)\right]_{q=q_c} \end{aligned} \tag{4.18}$$

where we have used the relations between the fields $\boldsymbol{E}$, $\boldsymbol{B}$ and the potentials ϕ and $\boldsymbol{A}$ in order to write down $\tau(t, D)$ in a gauge-independent way.

According to the values of $\boldsymbol{E}$ and $\boldsymbol{B}$ several behaviours may arise. We shall consider some examples.

4.3.2. (a) Static and uniform electromagnetic fields

$$\boldsymbol{E}(\boldsymbol{q}, t) = \boldsymbol{E}_0, \qquad \boldsymbol{B}(\boldsymbol{q}, t) = \boldsymbol{B}_0, \qquad \boldsymbol{E}_0, \boldsymbol{B}_0 \text{ constant}. \tag{4.19}$$

Now

$$\tau_{ij}(t, D) = m\{-\delta_{ij}D^2 + \epsilon_{ijk}\omega_0^k D\} \tag{4.20}$$

where $\boldsymbol{\omega}_0 \equiv e\boldsymbol{B}_0/mc$; the modulus of this vector, $\omega_0 \equiv |\boldsymbol{\omega}_0|$, is called the cyclotron frequency. Note that the constant electric field does not play any role in τ.

If $\omega_0 = 0$, $T_\Omega(\tau)$ is strictly positive for any domain Ω and the action A_Ω has a strict minimum at any q_c, i.e. any trajectory in a constant electric field is globally strictly minimal.

If $\omega_0 \neq 0$, let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be an orthonormal triad of real vectors such that $\boldsymbol{e}_3 \equiv \hat{\boldsymbol{\omega}}_0$. If we denote $\boldsymbol{e}^\pm \equiv 2^{-1/2}(\boldsymbol{e}_1 \pm \mathrm{i}\boldsymbol{e}_2)$, the operator $\tau(t, D)$ can be rewritten as

$$\tau_{jk}(t, D) = m\{-\delta_{jk}D^2 + \mathrm{i}\omega_0(e_j^+ e_k^{+*} - e_j^- e_k^{-*})D\}. \tag{4.21}$$

Therefore $T_\Omega(\tau)$ is the direct sum of three uncoupled operators, associated to the directions $\boldsymbol{e}^+$, $\boldsymbol{e}^-$, $\boldsymbol{e}_3$, namely

$$\tau^\pm(t, D) = -m(-D^2 \pm \mathrm{i}\omega_0 D), \qquad \tau_3(t, D) = -mD^2, \tag{4.22}$$

with Dirichlet boundary conditions at the end points of $\Omega =]t_1, t_2[$. Hence the lowest eigenvalue of $\tau(t, D)$ is

$$\mu_1(\Omega, \tau) = m\{(\pi/|\Omega|)^2 - \tfrac{1}{4}\omega_0^2\}. \tag{4.23}$$

Then, if we denote $T_0 \equiv 2\pi/\omega_0$, we have

(i) $|\Omega| < T_0 \Rightarrow A_\Omega$ has a strict minimum at q_c;

(ii) $|\Omega| > T_0 \Rightarrow A_\Omega$ is neither minimum nor maximum at q_c;

(iii) $|\Omega| = T_0 \Rightarrow A_\Omega$ is minimum at q_c (again the Lagrangian in this particular case of constant and uniform $\boldsymbol{E}$, $\boldsymbol{B}$ fields is quadratic).

4.3.2 (b) Circular motion in an attractive Coulomb potential

$$\boldsymbol{A} = 0, \qquad \phi(\boldsymbol{q}) = Q/|\boldsymbol{q}|, \qquad eQ < 0. \tag{4.24}$$

Now

$$\tau_{ij}(t, D) = -m\delta_{ij}D^2 + eQ|\boldsymbol{q}|^{-3}(\delta_{ij} - 3q_iq_j/|\boldsymbol{q}|^2)|_{q=q_c}. \tag{4.25}$$

For a circular orbit q_c of radius R_c, the eigenvalues of the matrix $eQ|\boldsymbol{q}|^{-3}(\delta_{ij} - 3q_iq_j/|\boldsymbol{q}|^2)$ evaluated at q_c are eQR_c^{-3}, double, and $-2eQR_c^{-3}$, simple. Consequently

$$\mu_1(\Omega, \tau) = m\pi^2/|\Omega|^2 + |eQ|R_c^{-3} \tag{4.26}$$

and therefore, remembering that the period of the circular orbit is $T_c = 2\pi(mR_c^3/|eQ|)^{1/2}$, we have

(i) $|\Omega| < \tfrac{1}{2}T_c \Rightarrow A_\Omega$ has a strict minimum at q_c;

(ii) $|\Omega| > \tfrac{1}{2}T_c \Rightarrow A_\Omega$ is neither minimum nor maximum at q_c.

Note that in both examples 4.3.2(a) and 4.3.2(b) the action A_Ω loses its character of minimum just when $|\Omega|$ is large enough that there is more than one possible trajectory connecting the end points.

4.3.3. Non-relativistic point particle in a one-dimensional potential

$$\mathscr{L} \equiv \tfrac{1}{2}\dot{q}^2 - V(q). \tag{4.27}$$

Again, the action integral A_Ω is locally strictly minimum at any q_c smooth on $\bar{\Omega}$, as discussed in § 4.2.1.

To determine whether A_Ω has a minimum (not only locally) at a given q_c, we must analyse the sign of the lowest eigenvalue of the operator $T_\Omega(\tau)$ defined in (3.15), which in this example is

$$T_\Omega(\tau) = -D^2 + W(t) \tag{4.28}$$

with Dirichlet boundary condition on $\partial\Omega$, $D \equiv -\mathrm{d}/\mathrm{d}t$ and

$$W(t) \equiv -V''(q_c(t)). \tag{4.29}$$

Fortunately, as we shall see in the following, some general conclusions can be reached without the precise knowledge of the trajectory $q_c(t)$, that in general is a non-trivial problem by itself.

4.3.3 (a) An anharmonic oscillator

$$V(q) = \tfrac{1}{2}kq^2 + \tfrac{1}{4}\beta q^4, \qquad k > 0, \beta \geq 0. \tag{4.30}$$

In this case $W(t) = -k - 3\beta q^2 \leq -k$. Therefore

$$T_\Omega(\tau) \leq -D^2 - k. \tag{4.31}$$

The lowest eigenvalue of the operator on the right-hand side of (4.31), with Dirichlet boundary conditions, is $\nu_1(\Omega) = \pi^2/|\Omega|^2 - k$. Therefore, a sufficient condition for $T_\Omega(\tau)$ to have at least one negative eigenvalue is $\nu_1 < 0$, i.e.

$$|\Omega| > \pi k^{-1/2}. \tag{4.32}$$

In consequence A_Ω has not an extremum at q_c if the interval $|\Omega|$ is large enough (a sufficient condition being (4.32)). Actually, a similar result holds for all potentials such that $V''(q) \geq 0$, $V''(q) \not\equiv 0$ (for instance, $V(q) = \beta q^{2n}$, $\beta > 0$); we omit the general proof for brevity.

4.3.3 (b) Some potentials with globally minimal trajectories. For all closed trajectories previously considered in § 4.3, the action integral loses its minimum character when the interval $|\Omega|$ is large enough. Nevertheless, this is not a general result, as we shall see in what follows.

Let us consider the double well potential

$$V(q) = -\tfrac{1}{2}\alpha q^2 + \tfrac{1}{4}\beta q^4, \qquad \alpha > 0, \beta > 0. \tag{4.33}$$

In this example $W(t) = \alpha - 3\beta q_c(t) > 0$ whenever $|q_c| < \alpha/3\beta$. Therefore it is to be expected that those trajectories that spend most of their time near the origin $q = 0$ will be globally minimal (for instance, in the limit $E = 0$ the particle takes an infinite time to reach the point $q = 0$, whereas the time spent in the $|q| > \alpha/3\beta$ region remains bounded, and so the behaviour approaches that for $W(t) > 0$ for all t). To confirm this hint in a rigorous way we shall prove the following theorem.

Theorem 4. Let $W(t)$ be the function defined in (4.29) and

$$S_\Omega \equiv \int_\Omega \mathrm{d}t\, W(t). \tag{4.34}$$

Then $S_\Omega > 0 \Rightarrow A_\Omega$ has a strict minimum at q_c.

Proof. For any $\gamma \in]0, 1[$ we can write down

$$T_\Omega(\tau) = -\gamma D^2 + (1-\gamma)\{-D^2 + (1-\gamma)^{-1} W(t)\}. \tag{4.35}$$

With the Dirichlet conditions, the first term on the RHS is strictly positive ($-\gamma D^2 \geq \gamma\pi^2/|\Omega|^2$). On the other hand, a beautiful theorem by Simon (1976) asserts that

$-\mathrm{d}^2/\mathrm{d}\xi^2 + V(\xi) \geqslant 0$ whenever $V(\xi)$ encloses a positive area with the real axis and tends to zero sufficiently fast when $|\xi| \to \infty$ (in our case, the variations vanish at the boundary $\partial\Omega$ and we extend $W(t) \equiv 0$, $\forall t \notin \Omega$). Therefore, $S_\Omega > 0 \Rightarrow T_\Omega(\tau) > 0 \Rightarrow A_\Omega$ strict minimum at q_c.

Furthermore: if the trajectory q_c is periodic with period T_c, then $S_\Omega > 0$ for some $\Omega = (t_1, t_1 + T_c) \Rightarrow q_c$ globally strictly minimal. The proof is straightforward: $S_{(t_1,t_1+T_c)} > 0 \Rightarrow S_{(t_1,t_1+nT_c)} = nS_{(t_1,t_1+T_c)} > 0$, $\forall n \Rightarrow A_{(t_1,t_1+nT_c)}$ strict minimum at $q_c \Rightarrow A_\Omega$ strict minimum at q_c, $\forall\Omega$, according to remark 3.

To apply this result to the case (4.33), we need only show the existence of trajectories for which $S_{(t_1,t_1+T_c)} > 0$. It is easy to prove that there is a region of energies around $E = 0$ for which this condition is verified (the boundaries of this region can be determined by solving a couple of transcendental algebraic equations involving elliptic functions, that we omit for brevity). For energies outside this region the action integral A_Ω is not minimum if the time interval $|\Omega|$ is large enough (this result was also to be expected: the potential (4.33) is approximately harmonic for energies near the minimum, $E_m = -\alpha^2/4\beta$, and it is approximately quartic for large energies, and as remarked in § 4.3.3(a), for potentials βq^{2n} the trajectories are not minimal if $|\Omega|$ is large enough).

5. Generalised Lagrangians

For those problems in which stationarity is not enough and minimality is needed, for instance in the search for solutions by variational methods, one might try to use a generalised (or weak) Lagrangian in the Ibragimov sense, such that any solution of a given equation (or system of equations) will be minimal with respect to the new action integral.

If $\omega[x, u(x)] = 0$ is the equation, the simplest choice is

$$\mathscr{L}[x, u(x)] = \tfrac{1}{2}\omega^2[x, u(x)]. \tag{5.1}$$

Clearly $A_\Omega[u_c + v] \geqslant 0 = A_\Omega[u_c]$, and hence A_Ω has a minimum at any u_c solution of $\omega[x, u(x)] = 0$, $\forall\Omega$. Obviously the action corresponding to (5.1) may have other critical points, namely the solutions of the associated Euler–Lagrange equation

$$\sum_\alpha (-1)^{|\alpha|} \frac{\delta\omega}{\delta u_\alpha} D^\alpha \omega = 0 \tag{5.2}$$

which do not satisfy $\omega = 0$.

However, with boundary conditions suitable for the original equation $\omega = 0$, (5.2) may have in general infinitely many solutions close to u_c, i.e. with vanishingly small stationary action. Thus from a practical viewpoint it may prove difficult to single out u_c, even in the case where u_c is strictly minimal for the new action.

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